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LETTER TO THE EDITOR

Fokker–Planck equation for time-dependent double-well potentials

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Abstract. An asymmetric potential with two local minima is considered. The time dependence is such that the true minimum at the initial stage is changed to the metastable minimum at the final stage. The redistribution of an initial probability configuration is studied for varying noise strengths and time scales for the change in the potential.

The quantum mechanical problem of tunnelling from one well to another as a double well goes through the transformation indicated in figure 1 has been studied (Berry 1984) and one finds that the limits $T \to \infty$ and $h \to 0$ (T fixed and large) lead to contrary behaviours of $|\psi(t)\rangle$ (where T is the time taken to carry through the transformation).

In this work, we describe a method to study the redistribution of probabilities for a double-well potential with a noise induced classical diffusion. Similar problems with a symmetrical double well (Schramm *et al* 1985) have been studied in optical (Scharpf *et al* 1987) and hydrodynamical (Swift *et al* 1991) contexts. Here we present the asymmetric case, which is the classical analogue of the quantum problem studied by Berry (1984).

The Fokker–Planck equation (for a review see Risken 1984) for a time-dependent potential $V(x, t)$ is

$$\frac{\partial P(x,t)}{\partial t} = \frac{\partial}{\partial x}\left[V'(x,t)P(x,t) + \varepsilon \frac{\partial P}{\partial x}(x,t) \right] \tag{1}$$

where ε is the noise strength.

Consider a potential $V(x, t)$ shown in figure 1, whose left well gets shallower and right well gets deeper until the configuration symmetric to the initial one is obtained. We assume that the probability distribution at zero time is given by the ground state of the potential as obtained from the Fokker–Planck equation. We will study the probability distribution in several cases that will be determined by T, the time taken for the transformation and ε, the noise term. We anticipate that in the limit $T \to \infty$ with finite ε and $\varepsilon \to \infty$ with finite T, the probability distribution should completely leak into the ground state of the final potential; whereas when $\varepsilon \to 0$, one expects that the initial configuration is maintained. The sudden case where T is very small is also studied and one can show that the leakage is determined by the noise term.

For analytical convenience, we model the potential by

$$V(x,t) = x^4 - \lambda(x - a(t))^2$$

where $a(t)$ goes from $+a$ to $-a$.

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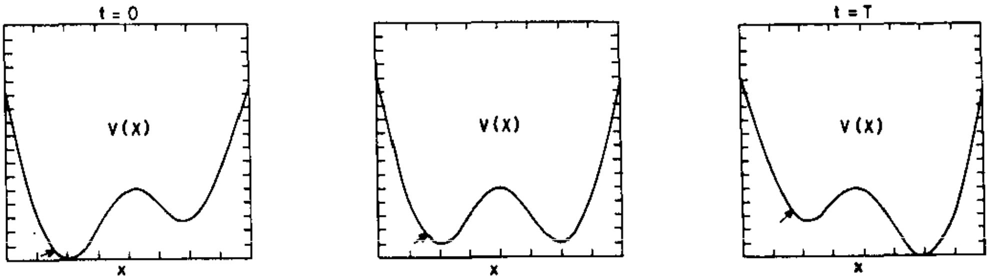

Figure 1. The time-dependent asymmetric double-well potential.

Then the initial probability distribution is

$$\psi_+ = N_+ \exp\left(-\frac{x^4+\lambda(x-a)^2}{\varepsilon}\right)$$

and the probability to be in the left well is much larger. In order to proceed we need another function which gives a greater probability for occupation of the right well and

$$\psi_- = N_- \exp\left(-\frac{x^4+\lambda(x+a)^2}{\varepsilon}\right)$$

is a reasonable guess. ψ_- is also important because it is the ground state of the final potential. Since the transition of the potential is smooth, we expect that at any time the probability distribution will be given, to a very good approximation, by

$$P(x, t) = \alpha(t)\psi_+ + \beta(t)\psi_-. \tag{2}$$

We are essentially interested in the time dependence of $\alpha(t)$ and $\beta(t)$. Inserting (2) into (1) we get

$$\dot{\alpha}\psi_+ + \dot{\beta}\psi_- = \frac{\mathrm{d}\psi_+}{\mathrm{d}x}(2\lambda b(t)) + \frac{\mathrm{d}\psi_-}{\mathrm{d}x}(2\lambda c(t))$$

where,

$$b(t) = a(t) - a \qquad \text{and} \qquad c(t) = a(t) + a.$$

Multiplying throughout by ψ_+ and integrating, we get

$$\dot{\alpha} + \gamma\dot{\beta} = \frac{2\lambda a}{\varepsilon}\gamma(2\lambda c(t))\beta. \tag{3}$$

Similarly for ψ_- we get

$$\gamma\dot{\alpha} + \dot{\beta} = -\frac{2\lambda a}{\varepsilon}\gamma(2\lambda b(t))\alpha \tag{4}$$

where,

$$(\psi_+, \psi_+) = (\psi_-, \psi_-) = 1$$

and

$$(\psi_+, \psi_-) = \gamma = \int_{-\infty}^{\infty} \mathrm{d}x \exp\left(\frac{-2x^4 + 2\lambda(x^2+a^2)}{\varepsilon}\right) N_+ N_-.$$

Writing this in matrix form and inverting we finally get

$$\begin{pmatrix}\dot{\alpha}\\ \dot{\beta}\end{pmatrix}=\frac{4\lambda^2 a\gamma}{\varepsilon(1-\gamma^2)}\begin{pmatrix}\gamma b(t) & c(t)\\ -b(t) & -\gamma c(t)\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}. \tag{5}$$

Equation (5) is our working equation and all the results of interest should be derivable from it.

A. Slow variation with a large noise term: (ε finite (but large) and $T\to\infty$). We note that $\gamma<1$. If $a\neq 0$, at large noise, both ψ_+ and ψ_- will spread and thus have considerable overlap. In such a case γ can be close to 1 and we can write $\gamma=1-\delta$ and work to first order in δ.

Equation (5) then reduces to

$$\begin{pmatrix}\dot{\alpha}\\ \dot{\beta}\end{pmatrix}=\frac{2\lambda^2 a}{\varepsilon\delta}\begin{pmatrix}(1-2\delta)b(t) & (1-\delta)c(t)\\ -(1-\delta)b(t) & -(1-2\delta)c(t)\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}.$$

Trying a solution of the form

$$\begin{pmatrix}\alpha\\ \beta\end{pmatrix}=\begin{pmatrix}\tilde{\alpha}\\ \tilde{\beta}\end{pmatrix}\exp\left(\int_0^t p(t')\,\mathrm{d}t'\right)$$

and neglecting $\dot{\tilde{\alpha}}$ and $\dot{\tilde{\beta}}$ (since the variation is slow) we get

$$\begin{aligned}p_1(t)&=\frac{2\lambda^2}{\varepsilon}(a^2-a^2(t))\\ p_2(t)&=-\frac{4\lambda^2 a^2}{\varepsilon\delta}+\frac{2\lambda^2}{\varepsilon}(3a^2+a^2(t)).\end{aligned} \tag{6}$$

If $a(t)$ is of the form $a(t/T)$ then one can write a general solution for the final probability distribution

$$\begin{pmatrix}\alpha\\ \beta\end{pmatrix}=r\begin{pmatrix}1\\ \dfrac{(1-2\delta)b(t)+1/a(a^2(t)-a^2)\delta}{(1-\delta)c(t)}\end{pmatrix}\exp\left(\frac{2\lambda^2}{\varepsilon}nT\right)$$
$$+s\begin{pmatrix}\dfrac{-(1-\delta)c}{(1-2\delta)b-p_2(\varepsilon\delta/2\lambda^2 a)}\\ 1\end{pmatrix}\exp\left(-\frac{4\lambda^2 a^2}{\varepsilon\delta}T+\frac{2\lambda^2}{\varepsilon}(4a^2)T-\frac{2\lambda_T^2}{\varepsilon}\right)$$

where

$$n=\frac{1}{T}\int_0^T(a^2-a^2(t))\,\mathrm{d}t>0$$

and r and s are determined by initial conditions.

We note that if $\delta<\frac{1}{2}$ only the first column vector contributes for large T. In that case,

$$\frac{\alpha}{\beta}(T\to\infty)=\lim_{T\to\infty}\frac{(1-\delta)c(T)}{(1-2\delta)b(T)}=0$$

i.e., given sufficient time and sufficiently large noise the probability distribution will leak into the right well.

B. (T finite and $\varepsilon \to \infty$). On the other hand, as $\varepsilon \to \infty$, one can show again that it is only the first column vector that is important.

For this we need to show that

$$\frac{4\lambda^2 a^2}{\varepsilon\delta} \to \infty \qquad \text{as } \varepsilon \to \infty.$$

Now, straightforward algebra for $\varepsilon \gg 1$ yields

$$\gamma = (\psi_+, \psi_-) \simeq 1 - \frac{8\lambda^2 a^2}{\varepsilon^2}\frac{V}{U}$$

where

$$V = \int_{-\infty}^{\infty} \left[\exp\left(-\frac{x^4 + \lambda x^2}{\varepsilon}\right)\right]^2 \mathrm{d}x \qquad U = \int_{-\infty}^{\infty} x^2 \left[\exp\left(-\frac{x^4 + \lambda x^2}{\varepsilon}\right)\right]^2 \mathrm{d}x$$

so that

$$\frac{4\lambda^2 a^2}{\varepsilon\delta} = \varepsilon \frac{U}{2V}.$$

Again it can be shown that for large, $V/U \sim \sqrt{\varepsilon}$ irrespective of λ. Therefore

$$\frac{4\lambda^2 a^2}{\varepsilon\delta} \text{ goes to infinity as } \varepsilon^{1/2}$$

so that once again we get

$$\left.\frac{\alpha}{\beta}\right|_{\text{final}} = 0$$

i.e. the probability distribution completely leaks into the ground state of the final potential.

C. (Low noise term ($\varepsilon \to 0$)). Now ε is small and we keep terms only up to first order in γ. The equation of interest is

$$\begin{pmatrix} \dot{\alpha} \\ \dot{\beta} \end{pmatrix} = \frac{4\lambda^2 a\gamma}{\varepsilon} \begin{pmatrix} 0 & c(t) \\ -b(t) & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

Again using the method described for case (A), we get

$$\frac{\alpha(t)}{\beta(t)} = \sqrt{\frac{a + a(t)}{a - a(t)}} \coth\left\{\frac{4\lambda^2 a\gamma}{\varepsilon} \int_0^t \sqrt{a^2 - a^2(t)}\,\mathrm{d}t\right\} \qquad \text{if } \alpha(0) = 1,\ \beta(0) = 0.$$

As $\varepsilon \to 0$, $\gamma/\varepsilon \to 0$, therefore $\beta(t)/\alpha(t) = 0$, i.e. there is no leakage to the ground state of the final potential.

D. ($T \to 0$). Last of all we consider the case where the change takes place very fast. We have equation (5)

$$\begin{pmatrix} \dot{\alpha} \\ \dot{\beta} \end{pmatrix} = \frac{4\lambda^2 a\gamma}{\varepsilon(1-\gamma^2)} \begin{pmatrix} \gamma b(t) & c(t) \\ -b(t) & -\gamma c(t) \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

Put $t = T\tau$.

After rescaling and letting T go to zero, we get (note that for $\varepsilon \ll 1$, $\gamma T/\varepsilon$ must be small and for $\varepsilon > 1$, T/δ must be small)

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} \to 0$$

as expected.

Thus we see that the qualitative features of noise induced classical diffusion are essentially the same as that of quantum tunnelling in the limits studied. Although the intermediate condition where T and ε are not at the extremes has not been presented, one can guess that there will be a mixed type of situation with incomplete leaking.

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